

# On optimality of the barrier strategy for a general Lévy risk process

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**Abstract** We consider the optimal dividend problem for the insurance risk process in a general Lévy process setting. The objective is to find a strategy which maximizes the expected total discounted dividends until the time of ruin. We give sufficient conditions under which the optimal strategy is of barrier type. In particular, we show that if the Lévy density is a completely monotone function, then the optimal dividend strategy is a barrier strategy. This approach was inspired by the work of Avram et al. (2007) [Annals of Applied Probability 17, 156-180], Loeffen (2008) [Annals of Applied Probability 18, 1669-1680] and Kyprianou et al. (2010) [Journal of Theoretical Probability 23, 547-564] in which the same problem was considered under the spectrally negative Lévy processes setting.

**Keywords** Lévy processes · Optimal dividend problem · Complete monotonicity · Barrier strategy · Scale function · Probability of ruin

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## 1 The model and problem setting

Let  $X = \{X_t : t \geq 0\}$  be a real-valued Lévy process define on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$  where  $\mathbb{F} = (\mathcal{F})_{t \geq 0}$  is generated by the process  $X$  and satisfies the usual

conditions of right-continuity and completeness. Denote by  $P_x$  the probability law of  $X$  when it starts at  $x$ . For notational convenience, we write  $P = P_0$ . Let  $E_x$  be the expectation operator associated with  $P_x$  with  $E = E_0$ . For  $\theta \in \mathbb{R}$ ,  $\kappa(\theta)$  denotes the characteristic exponent of  $X$  given by

$$\kappa(\theta) = \frac{1}{t} \log E(e^{i\theta X_t}) = ia\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x|<1\}}) \Pi(dx),$$

where  $a, \sigma$  are real constants, and  $\Pi$  is a positive measure supported on  $(-\infty, \infty) \setminus \{0\}$  which satisfies the integrability condition

$$\int_{-\infty}^{\infty} \min\{1, x^2\} \Pi(dx) < \infty.$$

The characteristics  $(a, \sigma^2, \Pi)$  are called the Lévy triplet of the process and completely determines its law;  $\Pi$  is called the Lévy measure; and  $\sigma$  is the Gaussian component of  $X$ . If  $\Pi(dx) = \pi(x)dx$ , then we call  $\pi$  the Lévy density. Such a Lévy process is of bounded variation if and only if  $\sigma = 0$  and  $\int_{-\infty}^{\infty} \min\{|x|, 1\} \Pi(dx) < \infty$ . It is well known that Lévy process  $X$  is a space-homogeneous strong Markov process.

For  $\mathbf{Re}\theta = 0$ , we can define the Laplace exponent of the process  $X$  by

$$\Psi(\theta) = \frac{1}{t} \log E(e^{\theta X_t}) = a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{\infty} (e^{\theta x} - 1 - \theta x \mathbf{1}_{\{|x|<1\}}) \Pi(dx). \quad (1.1)$$

That is,

$$E(e^{\theta X_t}) = e^{t\Psi(\theta)}, \quad \mathbf{Re}\theta = 0, \quad t \geq 0.$$

Using Ito's formula, we find that the infinitesimal generator for  $X$  is given by

$$\Gamma g(x) = \frac{1}{2}\sigma^2 g''(x) + ag'(x) + \int_{-\infty}^{\infty} [g(x+y) - g(x) - g'(x)y \mathbf{1}_{\{|y|<1\}}] \Pi(dy), \quad (1.2)$$

for  $g \in C^2$  with compact support, where  $\mathbf{1}_A$  is the indicator function of set  $A$ .

When  $\Pi\{(0, \infty)\} = 0$ , i.e., the Lévy process  $X$  with no positive jumps, is called the spectrally negative Lévy process. In this case, we recall from Bertoin (1998) and Kyprianou (2006) that for each  $q \geq 0$ , there exists a continuous and increasing function  $W^{(q)} : \mathbb{R} \rightarrow [0, \infty)$ , called the  $q$ -scale function defined in such a way that  $W^{(q)}(x) = 0$  for all  $x < 0$  and on  $[0, \infty)$  its Laplace transform is given by

$$\int_0^{\infty} e^{-\theta x} W^{(q)}(x) dx = \frac{1}{\Psi(\theta) - q}, \quad \theta > \phi(q),$$

where  $\phi(q) = \sup\{\theta \geq 0 : \Psi(\theta) = q\}$  is the right inverse of  $\Psi$ . Smoothness of the scale function is related to the smoothness of the underlying paths of the associated process. The following facts are taken from Kyprianou et al. (2010) and Chan et al. (2010). It is known that if  $X$  has paths of bounded variation then, for all  $q \geq 0$ ,  $W^{(q)}|_{(0, \infty)} \in C^1(0, \infty)$  if and only if  $\Pi$  has no atoms. In the case that  $X$  has paths of unbounded variation, it is known that, for all  $q \geq 0$ ,  $W^{(q)}|_{(0, \infty)} \in C^1(0, \infty)$ . Moreover, if  $\sigma > 0$ , then  $C^1(0, \infty)$  may be replaced by  $C^2(0, \infty)$ ; and if the Lévy measure has a density, then the scale function

is always differentiable. In particular, if  $\pi$  is completely monotone, then  $W^{(q)}|_{(0,\infty)} \in C^\infty(0,\infty)$ .

For general references on Lévy processes, we refer the reader, among others, to Bertoin (1998), Sato (1999) and Kyprianou (2006).

In this paper, we only consider the case that  $\Pi$  with density  $\pi$  is absolutely continuous with respect to Lebesgue measure. For the Lévy process  $X$ , we consider the following de Finetti's dividend problem. Let  $\xi = \{L_t^\xi : t \geq 0\}$  be a dividend strategy consisting of a left-continuous non-negative non-decreasing process adapted to the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  of  $X$ . Specifically,  $L_t^\xi$  represents the cumulative dividends paid out up to time  $t$  under the control  $\xi$  for an insurance company whose risk process is modelled by  $X$ . We define the controlled risk process  $U^\xi = \{U_t^\xi : t \geq 0\}$  by  $U_t^\xi = X_t - L_t^\xi$ . Let  $\tau^\xi = \inf\{t > 0 : U_t^\xi < 0\}$  be the ruin time when the dividend payments are taken into account. Define the value function of a dividend strategy  $\xi$  by

$$V_\xi(x) = E_x \left( \int_0^{\tau^\xi} e^{-\delta t} dL_t^\xi \right),$$

where  $\delta > 0$  is the discounted rate. The integral is understood pathwise in a Lebesgue-Stieltjes sense.

A dividend strategy is called admissible if  $L_{t+}^\xi - L_t^\xi \leq U_t^\xi$  for  $t < \tau^\xi$ . In words, the lump sum dividend payment is smaller than the size of the available capitals. Let  $\Xi$  be the set of all admissible dividend policies. The de Finetti dividend problem consists of solving the following stochastic control problem:

$$V_*(x) = \sup_{\xi \in \Xi} V_\xi(x),$$

and, if it exists, we want to find a strategy  $\xi_* \in \Xi$  such that  $V_{\xi_*}(x) = V_*(x)$  for all  $x \geq 0$ .

This optimization problem goes back to de Finetti (1957), who considered a discrete time random walk with step sizes  $\pm 1$  and showed that a certain barrier strategy maximizes expected discounted dividend payments. Optimal dividend problem has recently gained great attention in the actuarial literature. For the diffusion risk process, the optimal problem has been studied by many authors including Asmussen et al. (2000), Paulsen (2003), and Decamps and Villeneuve (2007). It is well known that under some reasonable assumptions, the optimality in the diffusion process setting is achieved by a barrier strategy (see, for example, Shreve et al. (1984)). The general problem for the Cramér-Lundberg risk model was first solved by Gerber in (1969) via a limit of an associated discrete problem. Recently, Azcue and Muler (2005) used the technique of stochastic control theory and Hamilton-Jacobi-Bellman (HJB) equation to solve the problem. They also included a general reinsurance strategy as another possible control. For the Cramér-Lundberg risk model with interest, Yuen et al. (2007) investigated some ruin problems in the presence of a constant dividend barrier; and Albrecher and Thonhauser (2008) derived the optimal dividend strategy which is again of band type and for exponential claim sizes collapses to a barrier strategy. In fact, for various risk models, many results in the literature indicate that a band strategy turns out to be optimal among all admissible strategies. For a more general risk process, namely the spectrally negative Lévy process, Avram et al. (2007)

gave a sufficient condition involving the generator of the Lévy process for the optimality of the barrier strategy; Loeffen (2008) connected the shape of the scale function to the existence of an optimal barrier strategy and showed that the optimal strategy is a barrier strategy if the Lévy measure has a completely monotone density; and Kyprianou et al. (2010) further investigated the optimal dividend control problem and showed that the problem is solved by a barrier strategy whenever the Lévy measure of a spectrally negative Lévy process has a density which is log-convex.

Motivated by the work of Avram et al. (2007), Loeffen (2008) and Kyprianou et al. (2010) for a spectrally negative Lévy process, our objective is to consider the optimal dividend problem for a general Lévy process (not necessarily spectrally negative). In contrast to the previous works, our approach does not rely on the theory of scale function of a spectrally negative Lévy process. Instead, the derivation of our results requires an introduction of a new function and the use of the Wiener-Hopf factorization theory. The rest of this paper is organized as follows. In Section 2, we present some preliminary results which are derived in Rogers (1983), Bertoin and Doney (1994) and Kyprianou (2006). In Section 3, we discuss a barrier strategy for dividend payments for the risk model of study. In Section 4, we give the main results and their proofs. Finally, Section 5 presents some examples and ends with a remark.

## 2 Two lemmas on the probability of ruin

In this section, we present two results on the probability of ruin which will be used later.

The first result is about the complete monotonicity of the probability of ruin. By definition, an infinitely differentiable function  $f \in (0, \infty) \rightarrow [0, \infty)$  is called completely monotone if  $(-1)^n f^{(n)}(x) \geq 0$  for all  $n = 0, 1, 2, \dots$ . Denote by  $\tau(q)$  an exponential random variable with mean  $1/q$  which is independent of the process  $X$ . For  $q = 0$ ,  $\tau(0)$  is understood to be infinite. Furthermore, let

$$\underline{X}_{\tau(q)} = \inf_{0 \leq t \leq \tau(q)} X_t \quad \text{and} \quad \overline{X}_{\tau(q)} = \sup_{0 \leq t \leq \tau(q)} X_t,$$

be the infimum and the supremum of the the Lévy process  $X$  killed at the random time  $\tau(q)$ , respectively. In the case with  $q = 0$ , we always assume that

$$E(X_1) = \Psi'(0) = a + \int_{-\infty}^{\infty} x \mathbf{1}_{\{|x| \geq 1\}} \pi(x) dx > 0. \quad (2.1)$$

We call the characteristic functions  $E(\exp(\alpha \underline{X}_{\tau(q)}))$  and  $E(\exp(\alpha \overline{X}_{\tau(q)}))$  the right and left Wiener-Hopf factors of  $X$ , respectively. According to Rogers (1983), the Wiener-Hopf factors are called mixtures of exponentials if there are probability measures  $H_+$  and  $H_-$  on  $(0, \infty]$  such that

$$E(\exp(\alpha \overline{X}_{\tau(q)})) = \int_0^{\infty} \frac{\lambda H_+(d\lambda)}{\lambda - \alpha}, \quad E(\exp(\alpha \underline{X}_{\tau(q)})) = \int_0^{\infty} \frac{\lambda H_-(d\lambda)}{\lambda + \alpha}.$$

Note that if  $E \exp(\alpha \underline{X}_{\tau(q)})$  is a mixture of exponentials, then the probability distribution function of  $\underline{X}_{\tau(q)}$  has the form

$$P_-(x) = \int_{(0,\infty]} e^{\lambda x} H_-(d\lambda), \quad x < 0.$$

Similarly, the probability distribution function of  $\overline{X}_{\tau(q)}$  has the form

$$P_+(x) = \int_{(0,\infty]} (1 - e^{-\lambda x}) H_+(d\lambda), \quad x > 0.$$

For  $x \geq 0$ , let  $\tau = \inf\{t \geq 0 : x + X(t) \leq 0\}$ . Define its Laplace transform and the probability of ruin as

$$\psi_q(x) = E(e^{-q\tau}), \quad \psi(x) = P(\tau < \infty),$$

respectively. Then,

$$\psi_q(x) = P(\underline{X}_{\tau(q)} \leq -x),$$

and

$$\psi(x) = \lim_{q \rightarrow 0} \psi_q(x) = P(\underline{X}_\infty \leq -x).$$

**Definition 2.1.** (Rogers (1983)) *The Lévy process  $X$  has completely monotone Lévy density if there exist measures  $\mu_+, \mu_-$  on  $(0, \infty)$  such that*

$$\pi(x) = \mathbf{1}_{\{x>0\}} \int_{(0,\infty)} e^{-tx} \mu_+(dt) + \mathbf{1}_{\{x<0\}} \int_{(0,\infty)} e^{tx} \mu_-(dt), \quad (2.2)$$

where

$$\int \frac{1}{t(1+t)^2} (\mu_+ + \mu_-)(dt) < \infty.$$

Theorem 2 of Rogers (1983) states that the jump measure  $\Pi$  has a completely monotone density if and only if the Wiener-Hopf factors of  $X$  are mixtures of exponential distributions. So, we have the following lemma.

**Lemma 2.1.** *If the jump measure  $\Pi$  has a completely monotone density, then the functions  $\psi_q(x)$  and  $\psi(x)$  are completely monotone in  $(0, \infty)$ .*

The second result is on Cramér's estimate for ruin probability. Roughly speaking, under suitable conditions, the probability of ruin decays exponentially when the initial capital becomes larger. The following lemma is extracted from Kyprianou (2006, Theorem 7.6); see also Bertoin and Doney (1994, Theorem).

**Lemma 2.2.** *Assume that  $X$  is a Lévy process which does not have monotone paths, for which*

(i) (2.1) holds;

(ii) there exists a  $R > 0$  such that  $\Psi(-R) = 0$  where  $\Psi$  is given by (1.1);

(iii) the support of  $\Pi$  is not lattice if  $\Pi(\mathbb{R}) < \infty$ .

Then,

$$\lim_{x \rightarrow \infty} e^{Rx} P(\tau < \infty) = \kappa(0, 0) \left( R \frac{\partial \kappa(0, \beta)}{\partial \beta} \Big|_{\beta=R} \right)^{-1},$$

where the limit is interpreted to be zero if the derivative on the right-hand side is infinite. For more details, see Kyprianou (2006, Theorem 7.6).

### 3 Barrier strategy

In this section, we consider a simple barrier strategy for dividend payments. Under a barrier strategy, if the controlled surplus reaches the level  $b$ , then the overflow will be paid as dividends; and if the surplus is less than  $b$ , then no dividends are paid out. Let  $\xi_b = \{L_t^b : t \geq 0\}$  be a barrier strategy and  $U^b = \{U_t^b : t \geq 0\}$  be the corresponding controlled risk process. Note that  $U_t^b = X_t - L_t^b$  and  $\xi_b \in \Xi$ . Moreover, if  $U_0^b \in [0, b]$ , then the strategy  $\xi_b$  corresponds to a reflection of the process  $X - b$  at its supremum; and if  $t \leq \tau_b = \inf\{t \geq 0 : U_t^b \leq 0\}$ , the process  $L_t^b$  can be defined by  $L_0^b = 0$  and  $L_t^b = \sup_{s \leq t} [X_s - b] \vee 0$ . Note that  $L_t^b$  is increasing, continuous and adapted such that the support of the measure  $dL_t^b$  is contained in the closure of the set  $\{t : U_t^b = b\}$ . If  $U_0^b = x > b$ ,  $L_t^b$  has a jump at  $t = 0$  of size  $x - b$  to bring  $U^b$  back to the level  $b$  and a similar structure afterward.

Let  $V_b(x)$  denote the dividend-value function if a barrier strategy with level  $b$  is applied. Then,

$$V_b(x) = E \left( \int_0^{\tau_b} e^{-\delta t} dL_t^b | U_0^b = x \right). \quad (3.1)$$

The following result shows that  $V_b(x)$  as a function of  $x$  satisfies the following integro-differential equations with certain boundary conditions. Note that Paulsen and Gjessing (1997) established a similar result for a very general jump-diffusion process.

**Theorem 3.1.** *Assume that the process  $X$  have Laplace exponent (1.1) and the infinitesimal generator  $\Gamma$  is given by (1.2). Let  $V_b(x)$  be bounded and twice continuously differentiable on  $(0, b)$  with a bounded first derivative and with the understanding that we mean the right-hand derivatives at  $x = 0$ .*

(i) If  $V_b(x)$  solves

$$\Gamma V_b(x) = \delta V_b(x), \quad 0 < x < b,$$

together with the boundary conditions

$$\begin{aligned} V_b(x) &= 0, & x < 0, \\ V_b(0) &= 0, & \text{if } \sigma^2 > 0, \\ V_b'(b) &= 1, \\ V_b(x) &= V_b(b) + x - b, & x > b, \end{aligned}$$

then  $V_b(x)$  is given by (3.1).

Furthermore, let  $\psi_b(x)$  be bounded and twice continuously differentiable on  $(0, b)$  with a bounded first derivative and with the understanding that we mean the right-hand derivatives at  $x = 0$ .

(ii) If  $\psi_b(x)$  solves

$$\Gamma\psi_b(x) = 0, \quad 0 < x < b,$$

together with the boundary conditions

$$\begin{aligned} \psi_b(x) &= 1, & x < 0, \\ \psi_b(0) &= 1, & \text{if } \sigma^2 > 0, \\ \psi'_b(b) &= 0, \end{aligned}$$

then  $\psi_b(x) = P_x(\tau_b < \infty)$ .

**Proof.** The proof of (i) can be done by using the arguments used in Paulsen and Gjessing (1997). If  $\sigma^2 > 0$ , the process starting from 0 immediately has a negative value. Hence,  $V_b(0) = 0$ . Applying Ito's formula to  $e^{-\delta(t \wedge \tau^b)} V_b(U_{t \wedge \tau^b}^b)$  gives

$$\begin{aligned} e^{-\delta(t \wedge \tau^b)} V_b(U_{t \wedge \tau^b}^b) &= V_b(x) + \int_0^{t \wedge \tau^b} e^{-\delta s} (\Gamma - \delta) V_b(U_s^b) ds \\ &\quad - \int_0^{t \wedge \tau^b} e^{-\delta s} V'_b(U_s^b) dL_s^b + M_t, \end{aligned}$$

where  $M_t$  is a martingale. Since  $V_b(U_{\tau^b}^b) = 0$  and the support of the measure  $dL_t^b$  is contained in the closure of the set  $\{t : U_t^b = b\}$ , taking expectation on both sides of the equality above yields

$$E \left( e^{-\delta(t \wedge \tau^b)} V_b(U_{t \wedge \tau^b}^b) \right) = V_b(x) - E \left( \int_0^{t \wedge \tau^b} e^{-\delta s} dL_s^b \right). \quad (3.2)$$

Therefore, (i) follows by letting  $t \rightarrow \infty$  in (3.2). The proof of (ii) is entirely analogous to the proof of (i).  $\square$

The integro-differential equation

$$\Gamma h(x) = \delta h(x), \quad x > 0,$$

has, apart from a constant factor, a unique nonnegative solution  $h(x)$ . This together with Theorem 3.1 gives

$$V_b(x) = \begin{cases} \frac{h(x)}{h'(b)}, & 0 \leq x \leq b, \\ x - a + \frac{h(b)}{h'(b)}, & x > b. \end{cases} \quad (3.3)$$

In particular, if  $\Pi\{(0, \infty)\} = 0$ , then  $h$  becomes the  $\delta$ -scale function  $W^{(\delta)}$  and (3.3) reduces to Proposition 1 of Avram et al. (2007), which was proved by the excursion theory. See also Renaud and Zhou (2007) and Zhou (2005) for an alternative approach.

## 4 Main results and proofs

Define a barrier level by

$$b^* = \sup\{b \geq 0 : h'(b) \leq h'(x) \text{ for all } x \geq 0\},$$

where  $h'(0)$  is understood to be the right-hand derivative at 0, and is not necessarily finite.

We now present the main results of the paper which concerns the optimal barrier strategy  $\xi_{b^*}$  for a general Lévy processes. This is a continuation of the work of Avram et al. (2007), Loeffen (2008), and Kyprianou et al. (2010) in which only the spectrally negative Lévy process was considered.

**Theorem 4.1.** *Suppose that  $h$  belongs to  $C^1(0, \infty)$  if  $\sigma = 0$  and  $\int_{-\infty}^{\infty} \min\{|x|, 1\} \Pi(dx) < \infty$  and otherwise belongs to  $C^2(0, \infty)$ . If  $\lim_{x \rightarrow \infty} h'(x) = \infty$  and  $h(x)$  is convex on  $[b^*, \infty)$ . Then, the barrier strategy at  $b^*$  is an optimal strategy.*

For simplicity, we write the Lévy density  $\pi$  as

$$\pi(x) = \begin{cases} \pi_1(x), & x > 0, \\ \pi_2(-x), & x < 0, \end{cases}$$

where  $\pi_1, \pi_2$  are Lévy measures concentrated on  $(0, \infty)$ .

**Theorem 4.2.** *If  $\pi_1$  and  $\pi_2$  are completely monotone on  $(0, \infty)$ , then the barrier strategy at  $b^*$  is an optimal strategy.*

Before proving the main results, we present several lemmas which are similar to those for spectrally negative Lévy process. For  $\delta > 0$ , we consider the following second order integro-differential equation:

$$\frac{1}{2}\sigma^2 h''(x) + ah'(x) + \int_{-\infty}^{\infty} [h(x+y) - h(x) - h'(x)y\mathbf{1}_{\{|y|<1\}}] \Pi(dy) = \delta h(x), \quad x > 0. \quad (4.1)$$

Set

$$\rho(\delta) = \sup\{\theta : \Psi(\theta) = \delta\}.$$

We assume that  $\rho(\delta) > 0$ . For such a  $\rho(\delta)$ , we denote by  $P^{\rho(\delta)}$  the exponential tilting of the measure  $P$  with Radom-Nikodym derivative

$$\left. \frac{dP^{\rho(\delta)}}{dP} \right|_{\mathbb{F}_t} = e^{\rho(\delta)X(t) - \delta t}.$$

Under the measure  $P^{\rho(\delta)}$ , the process  $X$  is still a Lévy process with Laplace exponent  $\psi_{\rho(\delta)}$  given by

$$\psi_{\rho(\delta)}(\eta) = \psi(\eta + \rho(\delta)) - \delta.$$

Let the process  $\tilde{X}$  has the Lévy triplet  $(\tilde{a}, \tilde{\sigma}^2, \tilde{\Pi})$  where  $\tilde{\sigma}^2 = \sigma^2$ ,  $\tilde{\Pi}(dx) = \tilde{\pi}(x)dx = e^{\rho(\delta)x}\pi(x)dx$ , and

$$\tilde{a} = a + \sigma^2 \rho(\delta) + \int_{-\infty}^{\infty} (e^{\rho(\delta)y} - 1)y\mathbf{1}_{\{|y| \leq 1\}} \pi(y)dy.$$



Moreover,

$$\int_{|x| \geq 1} e^{\rho(\delta)x} \pi(x) dx < \infty. \quad (4.2)$$

We refer the reader to Kyprianou (2006) for related discussions.

Note that the law of  $\tilde{X}$  is  $X$  under the new probability measure. Let  $\tilde{\psi}(x)$  be the ruin probability for the Lévy process  $\tilde{X}$ . Then, we have the following result.

**Lemma 4.1.** *The solutions of equation (4.1) are proportional to the function  $(1 - \tilde{\psi}(x))e^{\rho(\delta)x}$ .*

**Proof.** We first claim that  $E(\tilde{X}_1) = \tilde{\Psi}'(0)$  is always positive where  $\tilde{\Psi}$  is the Lévy exponent of  $\tilde{X}$ . In fact,

$$\begin{aligned} \tilde{\Psi}'(0) &= \tilde{a} + \int_{-\infty}^{-1} x \tilde{\pi} dx + \int_1^{\infty} x \tilde{\pi} dx \\ &= a + \rho(\delta)\sigma^2 + \int_{-\infty}^{\infty} y e^{\rho(\delta)y} \pi(y) dy - \int_{-1}^1 y \pi(y) dy \\ &= \Psi'(\rho(\delta)) > 0, \end{aligned}$$

since  $\Psi$  is strictly convex and increasing on  $[0, \rho(\delta)]$ . If  $\pi$  is completely monotone, then it follows from Definition 2.1 that  $\pi$  has the form (2.2). By using the integrability condition (4.2), we get  $\int_0^{\rho(\delta)} \mu_+(dt) = 0$ . Therefore,

$$\tilde{\pi}(x) \equiv e^{\rho(\delta)x} \pi(x) = \mathbf{1}_{\{x > 0\}} \int_{(\rho(\delta), \infty)} e^{-(t - \rho(\delta))x} \mu_+(dt) + \mathbf{1}_{\{x < 0\}} \int_{(0, \infty)} e^{(t + \rho(\delta))x} \mu_-(dt),$$

which shows that  $\tilde{\pi}$  is completely monotone. So, we see from Lemma 2.1 that  $\tilde{\psi}$  is also completely monotone, and hence  $\tilde{\psi} \in C^\infty(0, \infty)$ . Furthermore, it follows from Theorem 3.1 (ii) that  $\tilde{\psi}$  solves

$$\tilde{\Gamma} \tilde{\psi}(x) = 0, \quad x > 0,$$

together with the boundary conditions

$$\tilde{\psi}(x) = 1, \quad x < 0, \quad \text{and} \quad \tilde{\psi}(0) = 1 \quad \text{if} \quad \sigma^2 > 0,$$

where  $\tilde{\Gamma}$  is the infinitesimal generator for  $\tilde{X}$  given by

$$\tilde{\Gamma}g(x) = \frac{1}{2}\tilde{\sigma}^2 g''(x) + \tilde{a}g'(x) + \int_{-\infty}^{\infty} [g(x+y) - g(x) - g'(x)y\mathbf{1}_{\{|y| < 1\}}] \tilde{\pi}(y) dy.$$

Then, it can be shown by straightforward calculations that the function

$$h(x) = (1 - \tilde{\psi}(x))e^{\rho(\delta)x}$$

satisfies integro-differential equation (4.1). Hence, the result follows.  $\square$

Moreover, using arguments similar to those in Avram et al. (2007) and Loeffen (2008), we have the following two results.

**Lemma 4.2.** Suppose that  $h$  belongs to  $C^1(0, \infty)$  if  $\sigma = 0$  and  $\int_{-\infty}^{\infty} \min\{|x|, 1\} \Pi(dx) < \infty$ , and otherwise belongs to  $C^2(0, \infty)$ . If  $h$  is also convex on  $[b^*, \infty)$ , then

$$(\Gamma - \alpha)V_{b^*}(x) \leq 0, \quad \text{for } x > b^*.$$

**Lemma 4.3.** (Verification lemma) Suppose that  $\xi$  is an admissible dividend strategy such that  $V_\xi$  is twice continuously differentiable and for all  $x > 0$

$$\max\{(\Gamma - \alpha)V_\xi(x), 1 - V'_\xi(x)\} \leq 0.$$

Then,  $V_\xi(x) = V_*(x)$  for all  $x$ .

To end the section, we give the proofs of Theorems 4.1 and 4.2.

**Proof of Theorem 4.1.** The condition  $\lim_{x \rightarrow \infty} h'(x) = \infty$  implies that  $b^* < \infty$ . Clearly, it follows from the definition of  $V_{b^*}$  and Lemma 4.2 that for  $x > 0$

$$\begin{aligned} (\Gamma - \alpha)V_{b^*}(x) &\leq 0, \\ (1 - V'_{b^*}(x))(\Gamma - \alpha)V_{b^*}(x) &= 0, \\ V'_{b^*}(x) &\geq 1, \end{aligned}$$

which in turn imply that for all  $x > 0$

$$\max\{(\Gamma - \alpha)V_{b^*}(x), 1 - V'_{b^*}(x)\} = 0.$$

Hence, the result is a direct consequence of Lemma 4.3.  $\square$

**Proof of Theorem 4.2.** If  $\pi$  is completely monotone, then  $\tilde{\psi}$  is also completely monotone. Hence,  $\tilde{\psi}(x)$  admits the following representation

$$\tilde{\psi}(x) = \int_0^\infty e^{-sx} \mu(ds), \quad (4.3)$$

where  $\mu$  is a Borel measure on  $[0, \infty)$ . Making use of Cramér's estimate for ruin probability (see Lemma 2.2), we have

$$\tilde{\psi}(x) \sim C e^{-(R+\rho(\delta))x}, \quad x \rightarrow \infty,$$

where  $R$  is a positive constant such that  $\Psi(-R) = 0$ , and  $C$  is a nonnegative constant. Consequently, we have

$$\lim_{x \rightarrow \infty} e^{\rho(\delta)x} \tilde{\psi}(x) = 0.$$

This together with (4.3) give  $\int_0^{\rho(\delta)} \mu(ds) = 0$ , and hence

$$e^{\rho(\delta)x} \tilde{\psi}(x) = \int_{\rho(\delta)}^\infty e^{-sx + \rho(\delta)x} \mu(ds) = \int_0^\infty e^{-tx} \mu(\rho(\delta) + dt),$$

which is completely monotone. In particular, we have  $(e^{\rho(\delta)x} \tilde{\psi}(x))''' \leq 0$ . Thus,

$$h'''(x) = \rho(\delta)^3 e^{\rho(\delta)x} - (e^{\rho(\delta)x} \tilde{\psi}(x))''' > 0, \quad x > 0.$$

That is,  $h'$  is strictly convex on  $(0, \infty)$ . The result follows from Theorem 4.1 since  $\lim_{x \rightarrow \infty} h'(x) = \infty$ .  $\square$

## 5 Examples

In this section, we present an example with mixed-exponential jump-diffusion process and a list of completely monotone Lévy densities.

**Example 5.1.** (Mixed-exponential jump-diffusion process - see also Asmussen et al. (2004) and Mordecki (2004))

Consider the process  $X = \{X_t : t \geq 0\}$  given by

$$X_t = at + \sigma B_t + \sum_{k=1}^{N_t} Y_k, \quad (5.1)$$

where  $B = \{B_t : t \geq 0\}$  is a standard Brownian motion,  $N = \{N_t : t \geq 0\}$  is a Poisson process with parameter  $\lambda$ ,  $Y = \{Y_k : k \geq 1\}$  is a sequence of independent and identically distributed random variables with density

$$\pi(x) = \begin{cases} p \sum_{j=1}^m A_j \eta_j e^{-\eta_j x}, & x > 0, \\ q \sum_{j=1}^n B_j \varsigma_j e^{\varsigma_j x}, & x < 0, \end{cases} \quad (5.2)$$

with  $p, q \geq 0$ ,  $p + q = 1$ ,  $\eta_j, \varsigma_j > 0$ ,  $A_j, B_j \geq 0$ ,  $\sum_{j=1}^m A_j = 1$ , and  $\sum_{j=1}^n B_j = 1$ . As usual, we assume that the processes  $B$ ,  $N$ , and  $Y$  are independent. When  $m = n = 1$ , the process  $X$  reduces to the double-exponential jump-diffusion process (see Kou and Wang (2003)). Since  $\pi$  defined in (5.2) is completely monotone, a direct application of Theorem 4.2 shows that the barrier strategy at  $b^*$  is an optimal strategy.  $\square$

Besides Example 5.1, other examples of Lévy processes with completely monotone densities can be found in the literature. Several of them are listed below (for details, see Bagnoli and Bergstrom (2005), Loeffen (2008), Yin and Wang (2009) and Kyprianou, Rivero and Song (2010)):

- $\alpha$ -stable process with Lévy density:  $\pi(x) = \lambda x^{-1-\alpha}$ ,  $x > 0$  with  $\lambda > 0$  and  $\alpha \in (0, 1) \cup (1, 2)$ ;
- one-sided tempered stable process (particular cases include gamma process ( $\alpha = 0$ ) and inverse Gaussian process ( $\alpha = 1/2$ )) with Lévy density:  $\pi(x) = \lambda x^{-1-\alpha} e^{-\beta x}$ ,  $x > 0$  with  $\beta, \lambda > 0$  and  $-1 \leq \alpha < 2$ ;
- the associated parent process with Lévy density:  $\pi(x) = \lambda_1 x^{-1-\alpha} e^{-\beta x} + \lambda_2 x^{-2-\alpha} e^{-\beta x}$ ,  $x > 0$  with  $\lambda_1, \lambda_2 > 0$  and  $-1 \leq \alpha < 1$ .

Note that they all satisfy the condition  $\int_0^\infty \pi(x) dx = \infty$ . Moreover, some distributions with completely monotone density functions are given below:

- Weibull distribution with density:  $f(x) = c r x^{r-1} e^{-c x^r}$ ,  $x > 0$ , with  $c > 0$  and  $0 < r < 1$ ;

- Pareto distribution with density:  $f(x) = \alpha(1+x)^{-\alpha-1}$ ,  $x > 0$ , with  $\alpha > 0$ ;
- mixture of exponential densities:  $f(x) = \sum_{i=1}^n A_i \beta_i e^{-\beta_i x}$ ,  $x > 0$ , with  $A_i > 0$ ,  $\beta_i > 0$  for  $i = 1, 2, \dots, n$ , and  $\sum_{i=1}^n A_i = 1$ ;
- gamma distribution with density:

$$f(x) = \frac{x^{c-1} e^{-x/\beta}}{\Gamma(c) \beta^c}, \quad x > 0,$$

with  $\beta > 0$  and  $0 < c \leq 1$ .

**Remark.** For a spectrally negative Lévy process, whenever the Lévy measure has a density which is log-convex, then Kyprianou et al. (2010) showed that the optimal strategy is a barrier strategy. Since there is no condition on the upward jumps in Theorem 4.2, it might be possible to generalize the result of Kyprianou et al. (2010) to the present situation. That is, for any Lévy process with arbitrary positive jumps, if the Lévy density of negative jumps is log-convex, then the optimal dividend strategy is a barrier strategy. However, we are not able to give a formal proof of such a conjecture.  $\square$

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